ISRAEL JOURNAL OF MATHEMATICS, Vol. 37, No. 4, 1980

ESTIMATES OF THE INVARIANT DENSITIES OF ENDOMORPHISMS WITH INDIFFERENT FIXED POINTS

BY
MAXIMILIAN THALER

ABSTRACT

Numbertheoretical endomorphisms with indifferent fixed points are considered, whose invariant densities can be determined up to multiplication by functions bounded away from zero and infinity.

1. Introduction

A large number of recent publications deal with questions concerning absolutely continuous invariant measures of numbertheoretical endomorphisms. Most results stating sufficient conditions for the existence provide basic information about the measures and their densities. They may, for example, allow us to decide whether the invariant measures are finite or infinite, or to deduce analytic properties of the densities from the corresponding properties of the transformations (see References). The purpose of the present paper is to give estimates for densities belonging to transformations with infinite invariant measures, generalizing the estimates obtained by A. Rényi [9] in case of finite measures.

If a transformation satisfies Rényi's condition C, then the appertaining invariant density h fulfils

$$c_1 \leqq h(x) \leqq c_2.$$

where c_1, c_2 are positive constants. Many well-known examples which do not satisfy this condition have indifferent fixed points, and the invariant densities are not bounded. In this case it may be possible to find positive constants c_1, c_2 and a function h_0 such that

Received November 7, 1979

$$c_1 . h_0(x) \leqq h(x) \leqq c_2 . h_0(x),$$

where h_0 can be represented easily in terms of the transformation. Here we shall be dealing with such estimates, motivated by the growing interest of ergodic theorists in transformations preserving infinite measures.

The theory of jump transformations as developed by F. Schweiger in [10] and [11] turns out to be an appropriate tool to study this problem. Throughout the paper we make use of the results proved in [11], without repeating them in detail. In section 2 we prove the basic estimate. Section 3 contains applications and a few examples.

2. Definitions and main result

Let us first recall the definitions of indifferent fixed points and regular sources of a differentiable function $f : A \subseteq \mathbf{R} \to \mathbf{R}$.

$z \in \bar{A}$ is called an indifferent fixed point of f, if $\lim_{x \to z} f(x) = z$ and $\lim_{x \to z} |f'(x)| = 1$.

An indifferent fixed point is a regular source, if there exists a positive real α such that $|f'(x)|$ is decreasing on $(z - \alpha, z) \cap A$ and increasing on $(z, z + \alpha) \cap A$.

Let B be an interval with $(0,1) \subset B \subset [0,1]$, and let $\{B(i) : i \in I\}$ be a collection of non-degenerate disjoint subintervals whose union is B, where I contains at least two elements. We shall be concerned with transformations of the following type:

(T1) $T_i = T_{|B(i)}$ is differentiable, and $T' \geqq b > 0$ on B.

(T2) $TB(i) = B$; $\{B(i) : i \in I\}$ is a generator.

(T3) All indifferent fixed points are regular sources.

By (T1) and (T2) each $B(i)$ contains exactly one fixed point, which we denote by x_i.

(T4) The set $J = \{i \in I : x_i \text{ is indifferent fixed point}\}$ is non-empty and finite.

In particular, (B, T) is a fibered system in the sense of [11]. In accordance with our intention we suppose T admits a uniquely determined invariant measure absolutely continuous with respect to Lebesgue measure m. If the invariant measure is infinite, there is no natural way of normalizing the density. As our statements are not affected by the choice of the normalization we shall use the notation 'density' meaning a suitable version.

For each T, we define a jump transformation referred to in the sequel as the jump transformation associated with T. This definition requires further notations.

$$B(k_1,\cdots,k_n)=\bigcap_{i=1}^{n} T^{-i+1}B(k_i),\quad (k_1,\cdots,k_n)\in I^n,\quad n\geqq 1;$$

$$\mathscr{Z}=\text{class of all cylinders } B(k_1,\cdots,k_n),\quad n\geqq 1;$$

$$B_n(k)=B(\underbrace{k,\cdots,k}_{n\text{ times}}),\quad k\in I,\quad n\geqq 1;$$

$$\mathscr{D}_n=\{B_n(k):k\in J\},\quad n\geqq 1;\qquad \mathscr{A}=\mathscr{Z}\setminus\bigcup_{n=1}^{\infty}\mathscr{D}_n;$$

$$\mathscr{B}_n=\{B(k_1,\cdots,k_n)\in\mathscr{A}:B(k_1,\cdots,k_{n-1})\in\mathscr{D}_{n-1}\},\quad n\geqq 1;$$

$$D_n=\bigcup_{Z\in\mathscr{D}_n}Z;\qquad B_n=\bigcup_{Z\in\mathscr{B}_n}Z.$$

Our assumptions imply $\lim_{n\to\infty} m(D_n)=0$. Hence the jump transformation T^* given by $T^*(x)=T^n(x)$, $x\in B_n$, is defined a.e. on B. The choice of this particular class $\mathscr{A}$ as the starting point of the construction of T^* is motivated by the fact that jump transformations defined in this way frequently have invariant densities bounded away from 0 and ∞, and, in what follows, we shall demonstrate how to deduce the desired estimates from this information about T^*.

Denote the inverse of T_i by V_i, and define the functions $G_i(x)$, $F_i(x)$ in the following way:

$$G_i(x)=\begin{cases}(x-x_i).(Tx-x)^{-1}, & x\in B(i),\quad x\neq x_i,\\ 1, & x\in B\setminus B(i);\end{cases}$$

$$F_i(x)=\begin{cases}(x-x_i)(x-V_ix)^{-1}, & x\in B(i),\quad x\neq x_i,\\ 1, & x\in B\setminus B(i).\end{cases}$$

THEOREM 1. *Let T satisfy* (T1)–(T4), *and assume T^* to have an invariant density h^* satisfying $c_1^*\leqq h^*\leqq c_2^*$, where c_1^*, c_2^* are positive constants.*

Then there exist positive constants c_1, c_2 such that the invariant density h of T satisfies

$$c_1.\prod_{j\in J}G_j(x)\leqq h(x)\leqq c_2.\prod_{j\in J}F_j(x),\qquad x\in B\setminus\{x_j:j\in J\}.$$

First we prove an elementary lemma which is basic for all of our results.

LEMMA. *Let $a, e_i\in\mathbf{R}$ $(i=1,2)$, $e_i\geqq 0$, $e_1+e_2>0$, $E=(a-e_1,a+e_2)$, and let $f:E\to E$ be increasing and differentiable, satisfying*

(a) $|f(x)-a|<|x-a|$, $x \in E\backslash\{a\}$,

(b) f' *is increasing on* $(a-e_1, a)$ *and decreasing on* $(a, a+e_2)$.

Denoting by f_n *the* n*-th iterate of* f, *and putting* $g(x)=\sum_{n=1}^{\infty} f_n'(x)$, $x \in E$, *we have*

$$g(x) \leqq \{f(x)-a\}.\{x-f(x)\}^{-1}, \qquad x \in E\backslash\{a\}$$

and

$$g(x) \geqq \{x-a\}.\{f^{-1}(x)-x\}^{-1}, \qquad x \in f(E)\backslash\{a\}.$$

Proof. Suppose $e_2>0$, and define

$$g_N(t)=\sum_{n=1}^{N} f_n'(t), \qquad t \in (a, a+e_2).$$

Let $x \in (a, a+e_2)$ be fixed. We have $f(x)<x$, and

$$\int_{f(x)}^{x} g_N(t)dt=\sum_{n=1}^{N}[f_n(x)-f_{n+1}(x)]=f(x)-f_{N+1}(x).$$

By condition (b) f_n' is decreasing on $(a, a+e_2)$, so g_N is also decreasing. In particular,

$$g_N(f(x)) \geqq g_N(t) \geqq g_N(x) \qquad \text{for } f(x) \leqq t \leqq x.$$

Thus,

$$(x-f(x)).g_N(x) \leqq \int_{f(x)}^{x} g_N(t)dt=f(x)-f_{N+1}(x),$$

$$(x-f(x)).g_N(f(x)) \geqq \int_{f(x)}^{x} g_N(t)dt=f(x)-f_{N+1}(x).$$

Taking into account that $\lim_{n\to\infty} f_n(x)=a$, we obtain, letting $N \to \infty$,

$$(x-f(x)).g(x) \leqq f(x)-a,$$

$$(x-f(x)).g(f(x)) \geqq f(x)-a.$$

We replace x by $f^{-1}(x)$ in the second inequality, completing the proof in case $x>a$. The same kind of argument applies to the other case. □

Notice that equality holds for the special case $f(x)=a+q.(x-a)$, $0<q<1$.

Proof of the Theorem. The density h can be written in the form

$$h(x)=h^*(x)+\sum_{j \in J}\sum_{n=1}^{\infty} h^*(V_j^n x)(V_j^n)'(x)$$

(see [10], theorem 2; [11], theorems 4 and 5). In view of our assumptions this leads to

$$c_1^* \cdot \left[1+\sum_{j\in J}\sum_{n=1}^{\infty}(V_j^n)'(x)\right] \leqq h(x) \leqq c_2^* \cdot \left[1+\sum_{j\in J}\sum_{n=1}^{\infty}(V_j^n)'(x)\right].$$

By the conditions imposed on T, we can find neighbourhoods $E_j \subseteq B(j)$, $j \in J$, such that the functions $V_j : E_j \to E_j$ have the properties required in the Lemma.

In order to obtain the lower estimate we choose a positive number M_1 such that

$$\prod_{j\in J} G_j(x) \leqq M_1 \qquad \text{for } x \in B \setminus \bigcup_{j\in J} V_j(E_j).$$

Suppose first $x \in V_j(E_j)$, $x \neq x_j$, for some $j \in J$. Using the Lemma, we get

$$\begin{aligned} h(x) &\geqq c_1^* \cdot \sum_{n=1}^{\infty}(V_j^n)'(x) \\ &\geqq c_1^* \cdot (x-x_j)(T(x)-x)^{-1} \\ &= c_1^* \cdot \prod_{j\in J} G_j(x). \end{aligned}$$

For $x \in B \setminus \bigcup_{j\in J} V_j(E_j)$,

$$h(x) \geqq c_1^* \geqq c_1^* \cdot M_1^{-1} \cdot \prod_{j\in J} G_j(x).$$

Hence $c_1 = \min\{c_1^*, c_1^* \cdot M_1^{-1}\}$ is a suitable constant.

To prove the upper estimate we show first that the functions $\sum_{n=1}^{\infty}(V_j^n)'(x)$ are bounded on $B \setminus E_j$ for each $j \in J$. To do this we choose positive integers $p = p(j)$ such that $B_p(j) \subseteq E_j$. Using (T1) and applying the Lemma, we get

$$\begin{aligned} \sum_{n=1}^{\infty}(V_j^n)'(x) &= \sum_{n=1}^{p}(V_j^n)'(x) + \sum_{n=1}^{\infty}(V_j^n)'(V_j^p x)(V_j^p)'(x) \\ &\leqq \sum_{n=1}^{p} b^{-n} + b^{-p}\sum_{n=1}^{\infty}(V_j^n)'(V_j^p x) \\ &\leqq \sum_{n=1}^{p} b^{-n} + b^{-p}\{V_j^{p+1}(x)-x_j\}\cdot\{V_j^p(x)-V_j^{p+1}(x)\}^{-1}. \end{aligned}$$

Taking into account that $\{V_j(x)-x_j\}\cdot\{x-V_j(x)\}^{-1}$ is bounded on $B \setminus B_{2p}(j)$ and $V_j^p(x) \in B \setminus B_{2p}(j)$ for $x \in B \setminus E_j$, the statement follows.

Now let M_2, M_3 be positive real numbers such that

$$\sum_{\substack{k\in J\\ k\neq j}}\sum_{n=1}^{\infty}(V_k^n)'(x) \leqq M_2 \qquad \text{for } x \in E_j, \quad j \in J,$$

$$\sum_{j\in J}\sum_{n=1}^{\infty}(V_j^n)'(x)\leqq M_2 \qquad \text{for } x\in B\setminus\bigcup_{j\in J}E_j,$$

$$\prod_{j\in J}F_j(x)\geqq M_3 \qquad \text{for } x\in B\setminus\bigcup_{j\in J}E_j.$$

Then we have for $x\in E_j$, $x\neq x_j$,

$$\begin{aligned} h(x) &\leqq c_1^*\left[1+\sum_{n=1}^{\infty}(V_j^n)'(x)+\sum_{\substack{k\in J\\ k\neq j}}\sum_{n=1}^{\infty}(V_k^n)'(x)\right]\\ &\leqq c_1^*(1+M_2)(1+\{V_j(x)-x_j\}.\{x-V_j(x)\}^{-1})\\ &= c_1^*(1+M_2)(x-x_j)\{x-V_j(x)\}^{-1}\\ &= c_1^*(1+M_2)\prod_{j\in J}F_j(x). \end{aligned}$$

For $x\in B\setminus\bigcup_{j\in J}E_j$,

$$h(x)\leqq c_1^*(1+M_2)\leqq c_1^*(1+M_2).M_3^{-1}.\prod_{j\in J}F_j(x).$$

This completes the proof. □

If stronger differentiability conditions are imposed on T, the functions $\Pi_{j\in J}G_j$, $\Pi_{j\in J}F_j$ can be replaced by more simple ones. An important special case is considered in the following corollary.

COROLLARY 1. *Let T satisfy the conditions of the Theorem. Assume in addition that T admits expansions*

$$T(x)=x+a(j,n_j+1)(x-x_j)^{n_j+1}+a(n_j,n_j+2)(x-x_j)^{n_j+2}+\cdots,$$

$a(j,n_j+1)\neq 0$, at each indifferent fixed point x_j.

Then there exist constants $d_2\geqq d_1>0$ such that

$$d_1.\prod_{j\in J}|x-x_j|^{-n_j}\leqq h(x)\leqq d_2.\prod_{j\in J}|x-x_j|^{-n_j}.$$

PROOF. Epanding V_j at x_j yields

$$V_j(x)=x-a(j,n_j+1)(x-x_j)^{n_j+1}+b(j,n_j+2)(x-x_j)^{n_j+2}+\cdots$$

for x near x_j. Therefore,

$$|x-x_j|.|Tx-x|^{-1}=|x-x_j|^{-n_j}.R_j(x),$$

$$|x-x_j|\,.\,|x-V_j(x)|^{-1}=|x-x_j|^{-n_j}\,.\,r_j(x),$$

where the functions R_j, r_j are positive and bounded in a suitable neighbourhood of x_j, $j\in J$. The remaining arguments are similar to those in the preceeding proof, and are therefore omitted. □

3. Applications and examples

Before our results can be applied to a given transformation T, it must be shown that the associated jump transformation T^* has an invariant density bounded away from 0 and ∞. R. L. Adler has introduced a simple sufficient condition which ensures the existence of such a density for expansive transformations (see [2] and [3]). We shall prove that, in order to check this condition for T^*, it suffices to check it for T. Our results can therefore be applied to a large number of examples without much calculation.

The class $\bigcup_{n=1}^{\infty}\mathcal{B}_n$ is a suitable time-one-partition for T^*. As an index set I^* we take as usual

$$I^*=\{(k_1,\cdots,k_n):B(k_1,\cdots,k_n)\in\mathcal{B}_n,\ n\geqq 1\}.$$

If $k^*=(k_1,\cdots,k_n)\in I^*$, we denote the inverse of

$$T^*:B(k_1,\cdots,k_n)\to B$$

by W_{k^*}.

THEOREM 2. *Let T satisfy* (T1)–(T4). *Suppose the functions V_i $(i\in I)$ are twice differentiable, and*

$$A=\sup_{i\in I}\sup_{x\in B}|V_i''(x)|\,|V_i'(x)|^{-1}\quad\textit{is finite.}$$

Then

$$A^*=\sup_{k^*\in I^*}\sup_{x\in B}|W_{k^*}''(x)|\,|W_{k^*}'(x)|^{-1}\quad\textit{is finite.}$$

PROOF. Let $k^*=(k_1,\cdots,k_{N+1})\in I^*$ be fixed. It suffices to consider the case $N\geqq 1$.

In view of the definition of $\mathcal{B}_{N+1}$,

$$B(k_1,\cdots,k_{N+1})=B_N(j)\cap T^{-N}B(k)$$

for some $j\in J$ and some $k\in I$ with $k\neq j$. Therefore,

$$W_{k^*}(x)=V_j^N(V_k(x)).$$

It was shown in the proof of Theorem 1 that the functions $\sum_{n=1}^{\infty}(V_i^n)'(x)$ are bounded on $B\setminus B(i)$, $i\in J$. Hence, there exists a constant M such that

$$\sum_{n=1}^{\infty}(V_i^n)'(x)\leqq M \qquad \text{for all } x\in B\setminus B(i) \text{ and all } i\in J.$$

A standard computation yields

$$W_{k^*}''(x).[W_{k^*}'(x)]^{-1}=V_k'(x)\sum_{n=1}^{N-1}\{V_j''(V_j^n(V_kx))\}\{V_j'(V_j^n(V_kx))\}^{-1}(V_j^n)'(V_kx)$$
$$+V_k'(x)V_j''(V_kx)\{V_j'(V_kx)\}^{-1}+V_k''(x)\{V_k'(x)\}^{-1}.$$

From this we get

$$|W_{k^*}''(x)|.|W_{k^*}'(x)|^{-1}\leqq b^{-1}.A.\left[\sum_{n=1}^{N-1}(V_j^n)'(V_kx)+2\right]$$
$$\leqq b^{-1}.A.\left[\sum_{n=1}^{\infty}(V_j^n)'(V_kx)+2\right]$$
$$\leqq b^{-1}.A.(M+2),$$

where the last step is justified by $V_k(x)\in B\setminus B(j)$. □

Using Theorem 2 and Adler's Theorem we obtain the following sufficient condition for the results stated in section 2 to be applied.

COROLLARY 2. *Let T and V_i ($i\in I$) satisfy the conditions of Theorem 2. If T^* has a power with derivative bounded away from 1, then the hypotheses of Theorem 1 are fulfilled.*

In particular, T^* itself has the required property, if T' is bounded away from 1 on $B\setminus\bigcup_{j\in J}E_j$ for every choice of neighbourhoods E_j of x_j. □

It follows by theorem 2 in [11] that T is ergodic under the conditions of the Corollary. In this case, T is also conservative, so the invariant measure is uniquely determined.

In order to get some simple examples we consider a class of f-expansions.

Let $d\in\mathbf{N}\cup\{\infty\}$, $d\geqq 2$, and let $f:[0,d)\to[0,1)$ be increasing, twice differentiable and concave, with $f(x)<x$, $x\in(0,d)$, and $\lim_{x\to d}f(x)=1$. Assume further that $f''(x)\{f'(x)\}^{-1}$ is bounded. According to Rényi's terminology, the functions f determine f-expansions of type B with independent digits.

Applying Corollary 2 and the results of section 2 to $T(x)=f^{-1}(x)$ (mod 1) we conclude

$$c_1 . G(x) \leqq h(x) \leqq c_2 . F(x),$$

$$G(x) = \begin{cases} x . \{f^{-1}(x) - x\}^{-1}, & x \in (0, f(1)), \\ 1, & x \in [f(1), 1), \end{cases}$$

$$F(x) = \begin{cases} x . \{x - f(x)\}^{-1}, & x \in (0, f(1)), \\ 1, & x \in [f(1), 1), \end{cases}$$

resp.

$$d_1 . x^{-n} \leqq h(x) \leqq d_2 . x^{-n}, \qquad x \in (0,1), \quad d_2 \geqq d_1 > 0,$$

in case f admits an expansion $f(x) = x + a_{n+1}x^{n+1} + \cdots$, $a_{n+1} \neq 0$, valid in a neighbourhood of 0.

Notice that the first estimate is equivalent to

$$0 < \bar{c}_1 \leqq h(x) \leqq \bar{c}_2 \qquad \text{if } f'(0) < 1.$$

EXAMPLES

(1) Let us start with a few examples taken from elementary real analysis.

(1.1) $f(x) = x(1+x)^{-1}$;

(1.2) $f(x) = 1 - e^{-x}$;

(1.3) $f(x) = \tanh x$;

(1.4) $f(x) = (2/\pi)\arctan \pi x/2$.

By the preceeding arguments, the invariant densities h belonging to (1.1) and (1.2) can be written as

$$h(x) = x^{-1} . h_0(x), \qquad \text{where } 0 < c_1 \leqq h_0(x) \leqq c_2,$$

while the densities belonging to (1.3) and (1.4) are of the form

$$h(x) = x^{-2} . h_0(x), \qquad 0 < c_1 \leqq h_0(x) \leqq c_2.$$

It is well known that $h_0(x)$ is constant for example (1.1) (A. Rényi).

(2) Define

$$f_{n+1}(x) = x(x+1)[P_n(x)]^{-1/n}, \qquad x \geqq 0, \quad n \in \mathbf{N},$$

where $P_n(x) = \sum_{k=0}^{2n} a_k x^k$, $a_k = a_{n+k} = \binom{n}{k}$, $0 \leqq k \leqq n$. The functions f_{n+1} determine f-expansions satisfying the required conditions. As $f_{n+1}(x) = x - x^{n+2} \pm \cdots$ at 0,

$$h(x) = x^{-n-1} . h_0(x), \qquad 0 < c_1 \leqq h_0(x) \leqq c_2.$$

As a matter of fact, $h_0(x)$ is constant for these examples. They are special cases of the following continuous-parameter family of f-expansions.

Let $\alpha > 0$, $\alpha \neq 1$, and define

$$f_\alpha(x) = \{x^{1-\alpha} - (1+x)^{1-\alpha} + 1\}^{1/(1-\alpha)}, \qquad x > 0.$$

Since

$$f'_\alpha(x) = \{x^{-\alpha} - (1+x)^{-\alpha}\}.\{f_\alpha(x)\}^\alpha,$$

Kuzmin's equation

$$h_\alpha(x) = \sum_{k=0}^{\infty} h_\alpha(f_\alpha(x+k))f'_\alpha(x+k)$$

is easily checked for $h_\alpha(x) = x^{-\alpha}$. Each f_α has slope 1 at 0, so for $0 < \alpha < 1$ we have examples of transformations with indifferent fixed points and finite invariant measures.

It is interesting to note that Rényi's example (1.1) is obtained for $\alpha \to 1$.

(3) The transformations

$$Tx = x + x^{1+s} (\mathrm{mod}\ 1), \qquad 0 < s < 1,$$

mentioned in [4], show that $A < \infty$ is not necessary for $A^* < \infty$ (cf. Theorem 2).

Putting $A(x) = T''(x)\{T'(x)\}^{-2}$, we have

$$A(x) = s(1+s)x^{s-1}[1 + (s+1)x^s]^{-2},$$

hence $A = \sup_{x \in (0,1)} A(x) = \infty$.

Let $g(x) = x + x^{1+s}$, $x \in [0,1)$, $f = g^{-1}$, $B(0) = [0, f(1))$, $B(1) = [f(1), 1)$, and $q(x) = |f''(x)|\{f'(x)\}^{-1}$. As $g(x) = x + \{g(x)\}^{1+s}(1+x^s)^{-1-s}$, $f(x) = x - x^{1+s}.\bar{f}(x)$, where $\bar{f}(x)$ decreases from 1 to 2^{-1-s} on $[0,2]$.

$q(x) = A(f(x))$, so q is decreasing and integrable on $(0,1)$. We put $Q(x) = \int_0^x q(t)dt$. By exactly the same reasoning as in the proof of the Lemma,

$$\sum_{n=1}^{\infty} q(f^n(x))(f^n)'(x) \leqq Q(f(x))\{x - f(x)\}^{-1}, \qquad x \in (0,2).$$

Thus, adapting the formula for $W''_{k^*}(x).\{W'_{k^*}(x)\}^{-1}$ to this situation,

$$|W''_{k^*}(x)|\{W'_{k^*}(x)\}^{-1} \leqq Q(f^2(x+1))\{f(x+1) - f^2(x+1)\}^{-1} + q(f(x+1)) + q(x+1).$$

Since the right hand side is bounded, $A^* < \infty$.

For $s \geqq 1$, $A^* < \infty$ is implies by $A < \infty$. As $T'(x)$ is bounded away from 1 on $(e,1)$ for each $e > 0$, Corollary 2 applies, and we get by Theorem 1 for each $s > 0$,

$$c_1 . x^{-s} \leqq h(x) \leqq c_2 . x^{-s}, \qquad x \in (0,1), \quad c_2 \geqq c_1 > 0.$$

We conclude our considerations with an application to transformations of the entire real axis. Consider

$$T_n : \bar{\mathbf{R}} \rightarrow \bar{\mathbf{R}} : x \mapsto x - x^{-2n-1} \qquad (n \geqq 0).$$

By transforming on the unit interval via

$$x \mapsto 2^{-1}\{x(1+|x|)^{-1}+1\}, \qquad \text{for example,}$$

one gets a transformation with indifferent fixed points at 0 and 1, and Corollary 2 applies to the associated jump transformation (see also [10] for $n = 0$). Hence, looking at the expansions at 0 and 1, one sees that the density h_n of T_n can be represented as

$$h_n(x) = (1 + x^{2n}) . g_n(x),$$

where g_n is bounded away from 0 and ∞ on $\mathbf{R}$. As is well known, g_0 is constant.

Acknowledgements

The author would like to thank Prof. F. Schweiger for the introduction to this subject, and Dr. F. Österreicher for stimulating conversations.

This paper was written during a stay at the Mathematics Institute, University of Warwick. The author is greatly indebted to the Royal Society of London for granting a fellowship and to the Mathematics Institute of Warwick for its generous hospitality.

References

1. J. Aaronson, *Rational ergodicity and a metric invariant for Markov shifts*, Israel J. Math. **27** (1977), 93–123.

2. R. L. Adler, *F-expansions revisited*, in *Recent Advances in Topological Dynamics*, Lecture Notes in Math. **318**, Springer-Verlag, 1973.

3. R. Bowen, *Invariant measures for Markov maps of the interval*, Commun. Math. Phys. **69** (1979), 1–17.

4. M. Halfant, *Analytic properties of Rényi's invariant density*, Israel J. Math. **27** (1977), 1–20.

5. A. Lasota and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.

6. W. Parry, *Representations for real numbers*, Acta Math. Acad. Sci. Hungar. **15** (1964), 95–105.

7. P. Pianigiani, *Existence of invariant measures for piecewise continuous transformations*, to appear in Ann. Pol. Math.

8. G. Pianigiani, *First return map and invariant measures*, preprint.

9. A. Rényi, *Representations of real numbers and their ergodic properties*, Acta Math. Sci. Hung. **8** (1957), 477–493.

10. F. Schweiger, *Number theoretical endomorphisms with σ-finite invariant measures*, Israel J. Math. **21** (1975), 308–318.

11. F. Schweiger, *Some remarks on ergodicity and invariant measures*, Michigan Math. J. **22** (1975), 181–187.

12. F. Schweiger, *Lectures on fibered systems*, Mimeographed Notes, Salzburg 1977.

MATHEMATICS INSTITUTE
UNIVERSITY OF WARWICK
COVENTRY CV4 7AL, ENGLAND

AND

MATHEMATISCHES INSTITUT
UNIVERSITÄT SALZBURG
A-5020 SALZBURG, AUSTRIA